PLEASE ANSWER ALL QUESTIONS.
PLEASE EXPLAIN YOUR ANSWERS.

1. (a) Denote the normal-form game below by G. Solve G by iterated elimination of strictly dominated strategies. Explain briefly each step (1 sentence).

		Player 2		
		t_1	t_2	t_3
Player 1	s_1	$4, 8$	$6, 10$	$7, 6$
	s_2	$8, 4$	$4, 2$	$8, 0$
	s_3	$6, 7$	$12, 2$	$5, 4$
	s_4	$2, 8$	$9, 9$	$4, 10$

Solution: s_4 is dominated by s_3. After eliminating s_4, then t_3 is dominated by t_1. After eliminating t_3, then s_1 is dominated by s_3. After eliminating s_1, then t_2 is dominated by t_1. After eliminating t_2, then s_3 is dominated by s_2. Solution: (s_2, t_1).

(b) Suppose we repeat the stage game G twice. Denote the resulting game by $G(2)$. Find the set of pure-strategy Subgame-perfect Nash Equilibria of $G(2)$. Be careful to write out the equilibrium strategies.

Solution: Since there is a unique outcome of the iterated elimination of strictly dominated strategies, this is the unique NE of G. Hence, it must be played in every subgame of the finitely repeated game. $SPNE = \{(\text{play } (s_2, t_1) \text{ in every subgame})\}$.

(c) Consider now the infinitely repeated game with discount factor $\delta < 1$. Denote this game by $G(\delta)$. Is it possible to find a Subgame-perfect Nash Equilibrium of $G(\delta)$, for at least some values of $\delta < 1$, in which the average payoff of both players is strictly higher than their payoff in every Nash Equilibrium of G (i.e. of the 1-period game seen in part (a))? If so, find such an equilibrium. If not, argue why it does not exist. If you found such an equilibrium, be careful to argue why it is subgame perfect, and to show that neither player has an incentive to deviate from his equilibrium strategy.

Solution: Yes it is possible to find such a SPNE.

In the unique NE of the 1-period game, the players earn payoffs $(8, 4)$. One way they can both get a strictly higher payoff is if they play (s_4, t_2) in every period on the equilibrium path. We proceed by looking for a trigger strategy to support this, i.e. a strategy where (s_4, t_2) is played on the equilibrium path, and any deviation leads to (s_2, t_1) being played forever.

The trigger strategy is always optimal off the equilibrium path, since a stage-game NE is played each round. To show optimality on the equilibrium path, we observe that the problem is not symmetric, so we need to consider both players. For this solution, we use the definition of average payoffs from the lectures, ie. $(1-\delta)\sum \delta^{t-1}u_t$. Consider player 1. Playing s_4 on the equilibrium path yields average payoff 9. The best deviation on the equilibrium path is to play s_3 which yields 12 in the deviation round, and then 8 ever after. This gives an average payoff of $12(1-\delta)+\delta 8$. It is optimal not to deviate from the equilibrium strategy if

$$9 \geq 12(1-\delta) + 8\delta.$$

We can solve this for $\delta \geq 3/4$. Similarly, for player 2 not to deviate, we must have

$$9 \geq 10(1-\delta) + 4\delta.$$

This can be solved for $\delta \geq 1/6$. Taking the largest of these two values, the trigger strategy with (s_4, t_2) on the equilibrium path and (s_2, t_1) off the equilibrium path exists for $\delta \geq 3/4$, thus supporting average payoffs of 9 for both players.

2. Consider the extensive-form game given by the following game tree (the first payoff is that of player 1, the second payoff that of player 2):

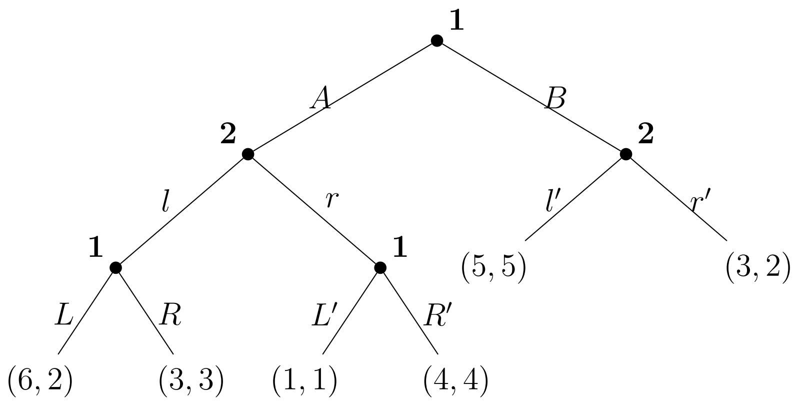

FIGURE 1

(a) How many proper subgames are there (excluding the game itself)? What are the strategy sets of the players?

Solution: 4 proper subgames. $S_1 = \{A, B\} \times \{L, R\} \times \{L', R'\}$. $S_2 = \{l, r\} \times \{l', r'\}$.

(b) Find all (pure strategy) Subgame-perfect Nash Equilibria.

Solution: Since this is a game of perfect and complete information, we can solve it by backward induction to get $SPNE = \{(B, L, R'), (r, l')\}$.

(c) Suppose now that player 1 does not observe the move of player 2, in situations where player 1 is called upon to move for a second time. That is to say, if player 1 chooses A, he does not observe whether player 2 then chooses l or r.

i. Draw the resulting game tree.

ii. Is this a game of perfect or imperfect information? How many proper subgames are there (excluding the game itself)? What are the strategy sets of the players?

iii. Show that there is a Subgame-Perfect Nash Equilibrium where player 1 has payoff 6. Briefly discuss why player 1 benefits from not being able to observe player 2's action (max. 3 sentences).

Solution: See Figure 2 for game tree. The game is of imperfect information. There are 2 proper subgames. Player 2's strategy set is as before, but now $S_1 = \{A, B\} \times \{L, R\}$. The subgame starting at player 2's choice between l and r can be written as.

		Player 2 l	Player 2 r
Player 1	L	$\underline{6}, \underline{2}$	$1, 1$
	R	$3, 3$	$\underline{4}, \underline{4}$

Thus, there are two NE of the subgame. We are looking for a SPNE with payoff 6 to player 1, so we take the NE (L, l). The right-hand side subgame is the same as before, with player 2 playing l'. Thus, player 1 has payoff 6 from playing A and payoff 5 from playing B. Therefore, $((A, L), (l, l'))$ is a SPNE and yields equilibrium payoff 6 to player 1. The unobservability of player 2's action implies that player 1 can credibly 'commit' to playing L if he believes that player 2 plays l. In this case, even if player 2 'deviates' and plays r, player 1 cannot observe this and will continue to play L. But this, in turn, makes it optimal for player 2 to play l.

3. Two tech entrepreneurs have made a gazillion dollar through a new app and need to decide how to allocate the gains. If they can't agree, nobody gets anything. Let x_1 and x_2 be the amounts that entrepreneur 1 and 2 get. Then their utilities are:

$$\begin{aligned} u_1(x_1) &= x_1 \\ u_2(x_2) &= 2\sqrt{x_2}. \end{aligned}$$

Find the Nash Bargaining Solution. What are the allocations?

Solution: We have $x_1 = v_1$ and $x_2 = v_2^2/4$. Hence $U = \{(v_1, v_2)|v_1, v_2 \geq 0, v_1 + v_2^2/4 \leq 1\}$. The Nash product becomes $v_1 v_2 = (1 - v_2^2/4)v_2$, where the equality comes from the usual efficiency restriction. First-order condition: $1 - 3v_2^2/4 = 0$. This gives $v_2^* = \frac{2}{\sqrt{3}}$. Then $v_1^* = 1 - (\frac{2}{\sqrt{3}})^2/4 = \frac{2}{3}$. The corresponding allocations are $x_1^* = \frac{2}{3}$ and $x_2^* = \frac{1}{3}$.

4. Suppose we are in a setting similar to the *common value auction* seen in class. There are two bidders, $i = 1, 2$. A single object is being sold in a **first-price auction**. Each bidder receives a uniformly distributed signal: for $i = 1, 2$,

$$s_i \sim u(0, 1).$$

Suppose that s_1 and s_2 are independent. The bidders' valuations depend on *both* signals, and are given by

$$v_1 = \frac{3}{4}s_1 + \frac{1}{4}s_2,$$
$$v_2 = \frac{3}{4}s_2 + \frac{1}{4}s_1.$$

Thus, the valuation of each bidder is 3/4 times his own signal plus 1/4 times his competitor's signal.

(a) What is the expected value of the object to the bidders *before* they enter the auction?

> **Solution:** Since both s_1 and s_2 have expectation 1/2, the unconditional expected value is 1/2. Conditional on his own signal, the expected value of bidder i is
> $$\frac{3}{4}s_i + \frac{1}{4}\mathbb{E}(s_j) = \frac{3}{4}s_i + \frac{1}{8}.$$

(b) Suppose the two bidders follow a symmetric linear strategy $\beta_i(s_i) = as_i$, where $a > 0$ is a positive constant. Conditional on s_1 and on winning the auction, what is player 1's expected value of the object? Explain why this is different to your answer in (a).

> **Solution:** If both players use the symmetric linear strategy, player 1's expected value of the object conditional on s_1 and on winning is
> $$\frac{3}{4}s_1 + \frac{1}{4}E(s_2|s_2 \leq s_1) = \frac{3}{4}s_1 + \frac{1}{4}\frac{s_1}{2} = \frac{7}{8}s_1 < \frac{3}{4}s_i + \frac{1}{8}.$$
> The reason is the 'winner's curse'.

(c) Show that there is an equilibrium with strategies of the form $\beta_i(\cdot)$, as seen in part (b). Explicitly solve for these equilibrium strategies i.e. find a.

> **Solution:** Since the problem is symmetric, we can focus on player 1. His expected profit from bidding b when his signal is s_1 is
> $$Pr(as_2 \leq b) \cdot \left[\frac{3}{4}s_1 + \frac{1}{4}E(s_2|as_2 \leq b) - b\right] = \frac{b}{a}\left[\frac{3}{4}s_1 + \frac{1}{4}\frac{b}{2a} - b\right].$$

The first-order condition with respect to b yields

$$\frac{1}{a}\left[\frac{3}{4}s_1 + 2\frac{1}{4}\frac{b}{2a} - 2b\right] = 0.$$

This can be solved for $b = \frac{3a}{8a-1}s_1$. The second-order condition is easy to check. So, matching coefficients we have $a = \frac{3a}{8a-1}$ which has solutions 0 and 1/2. Since we have assumed $a > 0$, then we are left with the unique solution $a = 1/2$.

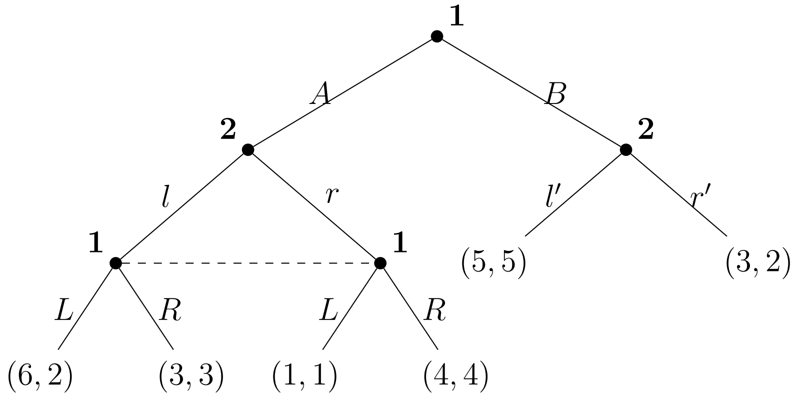

Figure 2